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Bäcklund equations on Kac-Moody-Lie algebras and integrable systems

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Abstract. *The introduction of an inverse sequence of Poisson spaces, determined by a Kac-Moody Lie algebra, allows to define completely integrable systems of P.D.E.'s described by the coadjoined of an inverse limit group and to extend the integrability techniques of M. Adler and P. van Moerbeke to systems of evolution equations. The P.D.E.'s appear when the flows are written along the integral curve of a fixed flow. This is achieved by means of a momentum operator which provides a geometrical description of the systems.*

INTRODUCTION

The main technique which is used by M. Adler and P. van Moerbeke to determine complete integrability is the description of the equations as the adjoint action flows on a graded Lie algebra, which is embedded in a Kac-Moody-Lie algebra. The integration is then obtained by the linearisation of the flows on the Jacobi variety of an algebraic curve. Completely integrable systems consisting of non linear evolution equations (of the KdV-type) have been studied by a growing list of people, see for example [1] [7] [9] [12] [14] [15] [16] [19] [22] [23]. The equations are written either in Lax form or in Zakharov-Shabat form; we will call the latter Bäcklund equations.

The main purpose of this paper is to show that one is able to extend the approach of M. Adler and P. van Moerbeke [2] to the latter systems.

This will be done in two steps. First one remarks that the introduction of an inverse sequence of Poisson spaces allows one to extend the finite dimensional systems of [2] into infinite dimensional ones. Next a jet bundle of C^∞-functions,

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or a subset of it, is considered as the area on which the evolution equations are described as special vector fields. A map defined on this space with values in the inverse sequence of Poisson spaces pulls back commuting flows, defining a commuting system of partial differential equations. This map will be called a momentum operator.

The paper is organized as follows. We study first the inverse sequence of Poisson spaces constructed from the Kostant-Adler-Symes theorem applied to a Kac-Moody-Lie algebra.

The next paragraph deals then with the definition of a momentum operator $\tilde{\sigma}$ describing P.D.E.'s on a subset S of a given function space $\mathscr{C}$. The only rôle of S is to restrict $\mathscr{C}$ such that the elements of S are mapped by $\tilde{\sigma}$ into the integral curves of a Hamiltonian vector field of the Poisson structures. We will call S the constraint equation. The inverse sequence structure of the space $\varprojlim \mathscr{K}_i$ determines by means of the operator $\tilde{\sigma}$ an inverse sequence structure on S: ... $\subset C^{i+1} \subset C^i \subset C^{i-1}$ ..., where the C^i are C^∞-integrable P.D.E.'s. It is this inverse sequence structure which will finally permit to determine the complete integrability of the equations.

Because any momentum operator $\tilde{\sigma}$ pulls back the ad-invariant forms from any $\mathscr{K}_i$ into constants, one finds that a given $\tilde{\sigma}$ fixed a specific orbit in each $\mathscr{K}_i$. Hence the momentum operators, on S, are parametrised by the ad-invariant forms on $\varprojlim \mathscr{K}_i$. For each value of these forms the operator $\tilde{\sigma}|_{C^i}$ determines a specific algebraic curve given by $\mathrm{Det}\,(\tilde{\sigma}(\lambda)|_{C^i} - z \cdot \mathrm{Id}) = 0$; λ is the grading parameter of $\varprojlim \mathscr{K}_i$. If on a given C^i the image of $\tilde{\sigma}$ statisfies the regularity conditions given in [2], one is able to determine a bijection of this image and a subset of the Jacobian of this curve. In these cases one may construct «finite zone» solutions [8]. We will not pursue this aspect in this paper.

The different momentum operators, which define a given evolution equation, determine different stratifications of solutions for this equation in the inverse limit. We also remark that the construction, as it is presented here, is purely local. Global extensions of the solutions may have obstructions.

The third paragraph then indicates how momentum operators are constructed in practice. Two basic lemmas are given together with some generalisations. It is clear that the existence of such operators is at the heart of this approach.

The last paragraph with examples shows the power of this approach in the construction of such systems. Examples of 1-parameter families of integrable systems are given as well as systems in more variables.

The problem of reduction of integrable systems, which is a very delicate question, becomes more transparent in this formulation. It is shown in the second paragraph that reductions can be obtained by choosing smaller orbits in the Poisson space, by taking different decompositions of the Lie algebra or by means of adjoint actions on the momentum operator.

The present approach is different from the other approaches in that the Poisson structure of the infinite dimensional Lie algebra is taken as the fundamental structure. The approach is very close to the work of Vinogradov and B. Kupershmidt [13] in that the jetbundle is considered as the appropriate space to describe commuting partial differential equations. This is in contrast with the work of A.G. Reyman and M.A. Semenov - T. Shansky [19], where some specific function space has to be selected to make the Lie algebra into a dynamical Lie algebra. The work of V.G. Drinfel'd and V.V. Sokolov [7] relies very heavily upon this approach. Our approach is in some ways very close to the work of G. Wilson [22]. One of the main lemmas is due to him. But is has to be remarked that the flows on the Lie algebra are not considered in the same way, nor is the approach of I.M. Gel'fand and L.A. Dikii used to describe complete integrability. Indeed it is the inverse limit space structure which allows one to define the exact number of equations needed at each grading to define integrability of one the flows in terms of some initial data.

A short sketch of the evolution of the ideas will help to situate our approach within the different approaches to this problem.

Initially we are interested in equations which can be written in the form

$$D_t A - D_x B + [A, B] = 0 \tag{1}$$

where A and B are $(n \times n)$-matrices with coëfficients in the ring of functions one some jetbundle J. D_t and D_x are total derivatives with respect to the coordinates (x, t) on an open subset of $\mathbb{R}^2$. These equations appear in the work of Wahlquist and Estabrook [21] and are very closely related to the formulation of a Bäcklund problem in the sense of Goursat [10] for a P.D.E. [16], [20], [4]. If a representation $\rho : gl(n, \mathbb{R}) \to \mathcal{X}(N)$ is choosen of $gl(n, \mathbb{R})$ in the Lie algebra of vector fields on a manifold N, with coordinates (v), one is led to the consideration of the system

$$v_x = (\rho \circ A)(v) \tag{2}$$

$$v_t = (\rho \circ B)(v). \tag{3}$$

Equation (1) is in some sense the compatibility condition for the system (2), (3). If one is able to eliminate the variables of the fibres of J from this system one is left with a P.D.E. on (v). The system (2), (3) is then a «Bäcklundtransformation» in the sense of Bäcklund-Bianchi-Clairin. As a consequence of the enormous interest in evolution equations in contemporary practices one prefers, roughly speaking, that equation (2) be a contact transformation. Equation (3) then becomes after elimination the modified equation of (1). The classical example is the KdV-equation written as (1). Equation (2) is then the Miura transformation and (3) is after elimination the MKdV-equation [6].

The famous Gardner transformation consists in the introduction of a parameter in (2), which allows one to invert this equation formally. This then provides an infinite set of conservation laws and symmetries, which are the basic ingredients for integration. It is this idea which is at the basis of our approach. The introduction of a parameter implies that equation (1) is written on a Kac-Moody-Lie algebra. The equations (2) and (3) become flows on this algebra. If one of the flows, for example equation (2), is inverted by analogy with the Gardner transformation, one obtains a map from a jet bundle into this Lie algebra. This map is a momentum operator which we define in the second paragraph.

Finally we remark that the theorems in the paper are not given in their greatest generality. This is done in order to keep the paper readable and to facilitate the insight in the constructions. As is shown in the examples, generalisations are easily obtained without fundamental alterations of the theorems and definitions.

Most theorems and propositions are written for maps in $C^\infty(\mathbb{R}^m, \mathbb{R}^p)$. Also the Lie algebras are mostly over $\mathbb{R}$. This restriction is not essential and an extension to the complex case is obvious.

1. INVERSE SEQUENCES OF POISSON SPACES

The setting of this paragraph is the Kostant-Adler-Symes (K.A.S.) theorem, which we will rephrase while defining the necessary concepts [2].

Let g be a finite dimensional Lie algebra equipped with a non degenerate ad-invariant bilinear form $(.\,,.)$ and let $g = k \dotplus n$ be a decomposition with k and n both Lie subalgebras. The subspace $k^\perp$ (resp. $n^\perp$) is the orthogonal subspace to k (resp. n) with respect to $(.\,,.)$. Hence $k^\perp$ is identified with the dual of n. By this identification the subspace $k^\perp$ inherits a Poisson structure given by the (Kirillov-Kostant) coadjoint action of n, coad (n), on $k^\perp$. This action is given by $p_{k^\perp}[p_n \nabla H, \xi]$, $\xi \in k^\perp, H \in \mathscr{F}(k^\perp)$ and ∇ the gradient defined by $(.\,,.)$. $p_{k^\perp}$ (resp. p_n) denotes the projection on $k^\perp$ (resp. n) defined by the decomposition $k^\perp \dotplus n^\perp$ (resp. $k \dotplus n$).

Let $W \subset k^\perp$ be a submanifold, invariant under coad (n) and let $\mathscr{A}(W)$ be the ring of C^∞-functios defined on a neighborhood of W, which are invariant under coad (g). Then $\mathscr{A}(W)$ is a system of commuting Hamiltonians on W. Moreover if H belong to $\mathscr{A}(W)$, then the corresponding Hamiltonian vector field is given by $[\xi, p_k \nabla H]|_W$, $\xi \in W$ and p_k the projection on k along n.

One remarks that $p_k \nabla H$ is a section of the k-vector bundle over $W \subset k^\perp$.

This theorem will be applied to decompositions of a Kac-Moody-Lie algebra which allows us to construct an inverse sequence.

A Kac-Moody-Lie algebra $\mathscr{L}$ constructed on g in given by

$$\mathscr{L} = \left\{ \sum_{-\infty}^{m} \xi_i \lambda^i \mid \xi_i \in g \text{ and } m \in \mathbb{Z} \text{ arbitrary} \right\}.$$

and equipped with the bracket $[\Sigma\, \xi_i \lambda^i, \Sigma\, \eta_j \lambda^j] = \sum_{i,j} [\xi_i, \eta_j] \lambda^{i+j}$ [2] [11] [18].

The bilinear form $(.\,,.)$ on g defines an infinite set of ad-invariant forms on h given by

$$(\xi, \eta)_\ell = \sum_{i+j=\ell} (\xi_i, \eta_j).$$

Each product $(\xi, \eta)_\ell$, for $\xi, \eta \in \mathscr{L}$, consists necessarily of a finite number of terms and hence is well defined. All ad-invariant polynomials (more specifically the trace forms) on g extend in the same way to $\mathscr{L}$. If $(.\,,.)$ is the Killing form on g, then $(.\,,.)_0$ is the Killing form on $\mathscr{L}$[2].

We define the truncated subspaces $\mathscr{L}^p$ by

$$\mathscr{L}^p = \left\{ \sum_{-\infty}^{p} \xi_i \lambda^i \mid \xi_i \in g \right\}.$$

The subalgebra of positive powers (resp. negative powers) in λ of $\mathscr{L}^p$ is denoted by $\mathscr{L}^p_+$ (resp. $\mathscr{L}^p_-$) and the zeroth order part by $\mathscr{L}^p_0$. Similar definitions are used for $\mathscr{L}$.

Multiplication by λ on $\mathscr{L}$ defines an ad-invariant map $\mathscr{L} \to \mathscr{L}$, namely $\lambda.\mathrm{ad}\xi = \mathrm{ad}\xi.\lambda$. This multiplication allows one to define the following limiting system

$$\longleftarrow \mathscr{L}^{p-1} \xleftarrow{\pi} \mathscr{L}^p \xleftarrow{\pi} \mathscr{L}^{p+1} \xleftarrow{\pi}$$

where the projection π stands for multiplication by λ^{-1}. The inverse limit $\overline{\mathscr{L}} = \varprojlim \mathscr{L}^p$ is obviously different from $\mathscr{L}$.

In applying the K.A.S.-theorem we will mainly be interested in the following type of decomposition satisfying:

(i) $\mathscr{L} = n \oplus k, \quad n \subset \mathscr{L}_- \oplus \mathscr{L}_0, \quad n \cap \mathscr{L}^{-p} = \mathscr{L}^{-p}$ for some $p \in \mathbb{N}$.

(ii) For any ad-invariant bilinear form of maximal rank $(.\,,.)_\ell$, the following inclusion is a strict inclusion:

$$k^{\perp p} \subset k^{\perp p+1}, \quad \text{with} \quad k^{\perp p} = k^\perp \cap \mathscr{L}^p, \quad p \in \mathbb{N}.$$

1.1. PROPOSITION. *Let $H_1, H_2 \in \mathscr{F}(k^{\perp p})$ and $\pi : k^{\perp p+1} \to k^{\perp p}$ the projection map. Then $\pi^*\{H_1, H_2\}_0 = \{\pi^* H_1, \pi^* H_2\}_1$, where $\{\cdot, \cdot\}_0$ (resp. $\{\cdot, \cdot\}_1$) is the Poisson bracket determined by $(\cdot, \cdot)_0$ (resp. by $(\cdot, \cdot)_1$ on the ring of functions on $k^{\perp p+1}/\ker \pi$).*

We denote $\bar{k}^{\perp} = \varprojlim k^{\perp p}$.

Proof. Let $(\cdot, \cdot)_0$ be given and define $n^{(p)}$, subspace of n, such that the form $(\cdot, \cdot)_0$ is non degenerate on $n^{(p)} \oplus k^p$. Let $k^{p\perp}$ and $n^{(p)\perp}$ be their orthogonals and ∇_0 (resp. ∇_1) the gradient with respect to $(\cdot, \cdot)_0$ (resp. $(\cdot, \cdot)_1$).

Then for $H_1, H_2 \in \mathscr{F}(k^{p\perp})$ one has

$$\begin{aligned} \{\pi^* H_1, \pi^* H_2\}_1 &= [\nabla_1 \pi^* H_1, \xi] \lrcorner \, \mathrm{d}\pi^* H_2, \qquad & \xi \in k^{p+1} \\ &= [\pi^* \nabla_0 H_1, \pi^* \xi'] \lrcorner \, \mathrm{d}\pi^* H_2, \qquad & \xi' \in k^p \end{aligned}$$

because π is ad-invariant. The restriction of ξ to $\pi^*\xi'$ has no influence, because $\nabla_0 H_1$ takes values in $n^{(p)}$ while $\pi^* H_2$ factors through $k^{\perp p+1}$ mod ker π_*. This implies that

$$\begin{aligned} \{\pi^* H_1, \pi^* H_2\}_1 &= \pi^*([\nabla_0 H_1, \xi'] \lrcorner \, \mathrm{d}H_2) \\ &= \pi^*\{H_1, H_2\}_0. \end{aligned}$$

To prove the proposition for any form $(. , .)_\varepsilon$ one remarks that this only shifts the vectorspaces orthogonal to n and k, which has no influnece on the proof. ■

There clearly exist other decompositions which give rise to inverse limit Poisson spaces. Some exampies will be given in the third paragraph. But the case described above will be the main situation dealt with in this paper. If n is chosen in this way there exists an infinite dimensional Lie group with Lie algebra n [2] [3].

On the space $\bar{k}^{\perp}$ the coordinates $\eta = (\eta_0, \eta_{-1}, \eta_{-2}, \ldots)$ will be used. For each $p \in \mathbb{N}$, one has the natural embedding:

$$j : k^{\perp p} \to \mathscr{L},$$

given by $\xi_p = \eta_0, \ \xi_{p-1} = \eta_{-1}, \ldots, \xi_{p-i} = \eta_{-i}, \ldots$. This identification is the natural one if one keeps in mind the inverse limit structure of $\bar{k}^{\perp}$. We will usually not mention this identification, but remark that the flows written as in the K.A.S. theorem are well defined on $\bar{k}^{\perp}$.

Let W be a submanifold of $\bar{k}^{\perp}$ invariant for coad (n), and ∇ the gradient operator with respect to $(\cdot, \cdot)_\varepsilon$.

1.2. PROPOSITION. *Let $H_1, H_2 \in \mathscr{A}(W)$, $W \subset k^{\perp p}$, for some p, and*

$$D_t \equiv [p_k \nabla H_1, \eta], \quad D_s \equiv [p_k \nabla H_2, \eta] \quad \text{on } W \subset \overline{k}^{\perp p}.$$

Then on W, one has:

(1) $$D_t p_k \nabla H_2 - D_s p_k \nabla H_1 + [p_k \nabla H_2, p_k \nabla H_1] = 0$$

(2) $D_t(\partial_\lambda p_k \nabla H_2, \eta)_m = D_s(\partial_\lambda p_k \nabla H_1, \eta)_m$, *with* ∂_λ *the partial derivative with respect to* λ *in* $\mathscr{L}$ *and* $m \in \mathbb{Z}$ *arbitrary.*

REMARK. (1) In this propostion $p_k \nabla H_i$ stands for $j^* p_k \nabla H_i$, which is well defined on each $k^{\perp p}$ and hence on $\overline{k}^{\perp}$.

(2) The relations (2) are defined for each bilinear ad-invariant form on $\mathscr{L}$. This form does not even have to be of maximal rank.

(3) If ∂ is an inner derivation, then with $\partial = \operatorname{ad} e_i$, $e_i \in \mathscr{L}$, one has for the terms in (2)

$$(\partial p_k \nabla H_1, \eta)_m = ([e_i, p_k \nabla H_1], \eta)_m$$
$$= (e_i, [p_k \nabla H_1, \eta])_m,$$

giving trivial relations coming from the components of D_s and D_t.

(4) The derivation with respect to λ, namely ∂_λ, on $\mathscr{L}$ is not an inner derivation. This is easily verified. Hence the derivation ∂_λ produces relations for (2) which do not necessarily come from the components of D_t and D_s. The term $(\partial_\lambda p_k \nabla H, \eta)_m$ is related to the central extensions of the algebra $\mathscr{L}$ [18].

Proof. Equation (1) is a direct consequence of the K.A.S. theorem and $\{H_1, H_2\} = 0$: The equation is obtained with the explicit use of the Poisson operator.

From (1) one obtains by derivation

$$D_t \partial_\lambda p_k \nabla H_2 - D_s \partial_\lambda p_k \nabla H_1 + [\partial_\lambda p_k \nabla H_2, p_k \nabla H_1] + [p_k \nabla H_1, \partial_\lambda p_k \nabla H_2] = 0.$$

Commutation of D_t (resp. D_s) with ∂_λ as derivations, on elements of the form $p_k \nabla H$, follows from the fact that the vector fields as sections of $T\mathscr{L}$ act as derivations on $\mathscr{F}(\mathscr{L})$ and hence on $p_k \nabla H$ as elements of a module, while ∂_λ is an operation on the fibre of the module. Taking the $(\cdot\,,\cdot)_m$ product with $\eta \in \overline{k}$, on has

$$(D_t \partial_\lambda p_k \nabla H_2, \eta)_m - (D_s \partial_\lambda p_k \nabla H_1, \eta)_m + ([\partial_\lambda p_k \nabla H_2, p_k \nabla H_1], \eta)_m + ([p_k \nabla H_2, \partial_\lambda p_k \nabla H_1], \eta)_m = 0.$$

One also has

$$(D_t \partial_\lambda p_k \nabla H_2, \eta)_m = D_t(\partial_\lambda p_k \nabla H_2, \eta)_m - (\partial_\lambda p_k \nabla H_2, D_t \eta)_m$$

and

$$(D_s \partial_\lambda p_k \nabla H_1, \eta)_m = D_s(\partial_\lambda p_k \nabla H_1, \eta)_m - (\partial_\lambda p_k \nabla H_1, D_s \eta)_m.$$

Using the expressions $D_t \eta = [p_k \nabla H_1, \eta]$ and $D_s \eta = [p_k \nabla H_2, \eta]$ and the ad-invariance of the form $(\cdot\,,\cdot)_m$, one obtains the desired relations. ∎

In the remainder of this paper we shall mainly be interested in the completely integrable Poisson structures obtained from $\mathscr{A}(W)$.

2. COMPLETELY INTEGRABLE SYSTEMS ON A JET BUNDLE

In order to write the Hamiltonian vector fields, on $\overline{k}^\perp$, over a jet bundle one needs a map σ from the jetbundle into $\overline{k}^\perp$ which allows one to pull back the vector fields into evolution equations or more generally into P.D.E.'s. This map σ, which we will call a momentum operator, has to respect the graded structure of $\overline{k}^\perp$. Hence σ must have a recurrence property and moreover the Hamiltonian vector fields in $\overline{k}^\perp$ have to be tangent to σ. Such a map will obviously not exist for each completely integrable system, but a large class seem to satisfy this condition. One hase in mind the KdV-system where the conservation laws reflect the inverse limit structure of a $\overline{k}^\perp$.

In this paragraph we will give a formulation for systems in one variable. In the fourth paragraph we will see how this formulation generalises to more variables and an example in three variables is presented.

All Poisson structures considered are purely local. Having the variable x on the line or on the circle is merely the choice of an appropriate function space which takes place on the level of the integration of a Cauchy problem.

Let J be a jet bundle, which is the space of the jets of germs of elements in $\mathscr{C} = C^\infty(\mathbb{R}, \mathbb{R}^m)$. The dimension m will be chosen large enough so that all operations are possible without the introduction of supplementary P.D.E.'s. In each concrete case this can be made very precise as will be seen. The variable on $\mathbb{R}$ will be x.

All P.D.E.'s will be supposed to be locally C^∞-integrable and will always be identified with their prolonged submanifolds in J.

2.1. DEFINITION. *Let $\sigma : J \to \overline{k}^\perp$ be a map such that*

(1) $\exists \{C^i\}, i \in \mathbb{N}$ *all P.D.E.'s such that*

(a) $C^i \subsetneq C^{i+1}, \forall i$

(b) $\sigma|_{C^i} : C^i \to \bar{k}^\perp$ *is a smooth map with values in a finite-dimensional truncated subspace, and* $\pi \circ \sigma|_{C^i} = \sigma|_{C^{i-1}}$, $\forall i$.

(2) $\forall r$, $\forall H_i \in \mathscr{F}(\bar{k}^{\perp r})$, *the vector fields* $p_{k^\perp}[p_n \nabla H_i, \eta]|_\sigma$ *are tangent to* σ.

Then σ *is a momentum operator defining a differential operator* $\tilde{\sigma} : \mathscr{C} \to \mathbb{C}^\infty(\mathbb{R}, \bar{k}^\perp)$.

2.2. DEFINITION. *Let* σ *be a momentum operator and* $S \subset J$ *a P.D.E.; then* σ *is holonomic if for each* r $\exists H_0 \in \mathscr{F}(\bar{k}^{\perp r})$ *such that on* S

$$D_x \sigma = P_{k^\perp}[\sigma^* p_n \nabla H_0, \sigma].$$

The equations S will be called the constraint equation and the C^i, the truncating equations. One remarks that from the definition it follows that the set $\{C^i\}$ defines invariant submanifolds of $\mathscr{C}$ for the system defined by $\tilde{\sigma}$. The same is true for the equation S.

The set $\{C^i\}$ defines on J an inverse sequence structure reflecting the structure of $\bar{k}^\perp$, while S restricts the set $\mathscr{C}$ such that all elements in S are mapped into the integral curves of the Hamiltonian vectorfield $p_k[p_n \nabla H_0, \eta]$.

This implies that the parameter t_0 along the integral curves of this field coincides with the variable x.

REMARKS. (a) For pratical reasons we will choose a trace form of rank two for the function H_0.

(b) If $H_0 = (\xi, \xi)_r$, $\xi \in \mathscr{L}^p$ one has

$$\nabla_\ell H_0 = \nabla_0 (\xi, \xi)_{r+\ell},$$

where ∇_ℓ (resp. ∇_0) is the gradient with respect to $(\cdot\,,\cdot)_\ell$ (resp. $(\cdot\,,\cdot)_0$). This allows us to choose $(\cdot\,,\cdot)_0$ to define the orthogonals and the gradient on $\mathscr{L}$ for the rest of this approach.

To facilitate the discussion we impose some conditions on the decompositions we consider.

Condition (C). Let $\mathscr{E}_0 = n \cap \mathscr{L}_0$ and $\mathscr{E}_{-1} = n \cap \mathscr{L}^{-1}/\mathscr{L}^{-2}$; then n satisfies condition (C) if

(1) $n \cap \mathscr{L}^{-2} = \mathscr{L}^{-2}$

(2) $\mathscr{E}_0 \neq \mathscr{L}_0$

(3) $\mathscr{E}_{-1} \neq \bar{0}$.

Now for a fixed p let the Hamiltonian function H_0 be $(\xi, \xi)_{p-1}$ on $\mathscr{L}^p$. Using the coordinates $(\eta_0, \eta_{-1}, \ldots)$ and the notation $\sigma^*\eta_{-1} = \sigma_{-1}$ one has

$$\sigma^* p_k \nabla H_0 = p_k(\sigma_{-2}\lambda^{-1} + \sigma_{-1} + \sigma_0 \lambda).$$

We remark that the projection p_k is a projection in the fibre of the k-vector-bundle over $\bar{k}^{\perp}$.

Now let $\Psi : J \to k$ be a smooth map with values in $k \cap \mathscr{L}^1$ and let $\sigma : J \to \bar{k}^{\perp}$ be a smooth map, solution of the equation

(i) $$D_x \sigma = [\Psi, \sigma].$$

Then the equation $S : \sigma^* p_k \nabla H_0 = \Psi$ is a P.D.E. such that for each integrable local section Jf in S, $f \in \mathscr{C}$, the map $\sigma \circ Jf$ is an integral curve of the vectorfield $[p_k \nabla H_0, \eta]$ on $\bar{k}^{\perp}$.

2.3. THEOREM. *Let Ψ, σ and $H_i \in \mathscr{A}(\bar{k}^{\perp p})$ for some p, be as above.*

(1) *The following P.D.E.'s are equivalent on S*

(a) $D_{t_i}\Psi - D_x \sigma^* p_k \nabla H_i + [\Psi, \sigma^* p_k H_i] = 0$

(b) $D_{t_i}\sigma = [\sigma^* p_k \nabla H_i, \sigma]$

(2) *The equations*

(c) $D_{t_i}(\partial_\lambda \Psi, \sigma)_m = D_x(\partial_\lambda \sigma^* p_k \nabla H_i, \sigma)_m$

are identically satisfied on (a).

Proof. The equations $D_x \sigma = [\Psi, \sigma]$ are satisfied identically on S. Hence from prop. (1-2) it follows that the set (b) imply the equations (a). Moreover on S one has $p_k(\sigma_{-2}\lambda^{-1} + \sigma_{-1} + \sigma_0 \lambda) = \psi$. Because multiplication by λ is ad-invariant, the set (a) is a subset of (b). Now the proof of part (1) follows from the construction of σ, which will be given in paragraph three; namely each σ_{-k} depends algebraically on Ψ and a finite number of D_x derivatives of Ψ. Because all D_{t_i} commute with D_x on (a) all other equations in (b) are prolongations of (a). This proves part (1).

Part (2) follows from proposition (1-2) and part (1).

REMARK. (1) The theorem has a generalisation for non holonomic momentum operators. For the proof of this one has to rely more heavily upon the construction of σ.

(2) The equations (c), for a fixed i, are elements of the total cohomology of equations (a). It is sufficient to write

$$\theta_m \equiv (\partial_\lambda \Psi, \sigma)_m \, dx + (\partial_\lambda \sigma^* \nabla H_i, \sigma) \, dt_i.$$

Then on (a) one has $d_H \theta_m = 0$, where d_H is the total de Rham operator on equation (a) [5].

(3) If $\partial_\lambda \psi$ is constant it follows from the previous theorem that system (a) has a subset of equations which are conservation law type equations.
The resulting equations in (a) depend on the choice of Ψ. To make sure that this construction does not imply supplementary equations besides (a) we need some more criteria avoiding this.

(4) One verifies that ∂_λ may be replaced by any derivation commuting with D_x and the D_{t_i}, for example D_x or any D_{t_i}. See for example also [19]. More specifically any symmetry vector field of equations (a) will do.

2.4. DEFINITION. *A set of functions* $\{\phi_i\}$, $i = 1, \ldots, \ell$, *on a jet bundle, is functionally totally independent (F.T.I.) if each* ϕ_i *is functionally independent of all other* ϕ_i *together with their total derivatives up to any order.*

2.5. DEFINITION. *A map* $f : J \to N$, *with* N *an affine space, is smooth free map if*

(1) *f is smooth*

(2) *f is vertical on J. (f is independent of the base coordinate x)*

(3) Im (f) *is an affine subspace W of N*

(4) *the components of f with respect to any base in W are F.T.I.*

2.6. THEOREM. *Let* $\Psi : J \to k \cap \mathscr{L}^1$ *be a smooth free map; then the P.D.E.'s of* (1.a) *in theorem* (2.3) *are independent, i.e. the equations do not imply any supplementary P.D.E.'s on* $S \subset J$.

The proof of this theorem is straightforward.

Let σ be a momentum operator, holonomic on S, one then easily derives the following proposition.

2.7. PROPOSITION. *Let g be a real split semisimple algebra. Then a set of necessary conditions for* σ *to be a momentum operator is given by*

$$\widetilde{Q}_i(\sigma) = \text{constant}, \quad i = 1, \ldots, \ell,$$

where $\widetilde{Q}_i$ *are the ad-invariant forms on* $\mathscr{L}$ *constructed from the* Q_i *on g (rank* $g = \ell$).

Consequently one finds that any σ, solution of $D_x \sigma = [\Psi, \sigma]$, on $S : \Psi = = \sigma^* p_k \nabla H_0$, depends on the ad-invariant forms $\widetilde{Q}_i$ in a parametric way. In other terms: a given σ selects a specific orbit of the *coad* (n)-action on W. A different choice of these parameters defines a different inverse sequence structure $\{C^i\}$ on S.

The selection of those σ's which define the same $D_{t_i}\sigma = [\sigma^* p_k \nabla H_i, \sigma]$, for a given H_i, determines different stratifications of solutions of this equation. For the subsequent we will assume that g is a real split semisimple algebra, unless stated otherwise.

One of the main problems in completely integrable systems is the problem of reduction. Reductions of the equations appear in this approach on three levels, namely by taking a different decomposition, by selecting different coad (n)-invariant subspaces W in $\overline{k}^{\perp}$ and by taking transformations of the momentum operator and thus changing the constraint equation S. We will give a short introduction to these three possibilities.

(i) The selection of coad (n)-invariant subspaces W in $\overline{k}^{\perp}$. The map σ has to take values in W and hence by selecting W one imposes conditions on σ as well as on W.

From Prop. (2.7) we obtain the conditions $(\eta_0, \eta_0) =$ constant and $(\eta_0, \eta_{\ 1}) =$ = constant, defining coad (n)-invariant subspaces of $\overline{k}^{\perp}$ which will be used frequently in the examples.

The following proposition allows one to select further submanifolds in $\overline{k}^{\perp}$. The proof follows from the action $p_{k^{\perp}}[p_n \nabla H, \eta]$ on $\overline{k}^{\perp}$.

2.8. PROPOSITION. *Let V be a vector subspace of $\overline{k}^{\perp}$, which is given by $F(\eta_0) = 0$ with F a linear function. Then V is a* coad (n)-*invariant subspace of $\overline{k}^{\perp}$ iff* Im ad $\mathscr{E}_0(V) = V$.

If we restrict attention to the case dim $V = 1$ we distinguish two subcases.

case α : V is a regular semi simple subspace of g.

case β : V is the 1-dimensional root subspace $g_{-\beta}$, according to a root space decomposition of g and β being the hithest root.

Both are clarly already reduction of the general case $(\eta_0, \eta_0) =$ constant.

2.9. PROPOSITION. *Let V be as in the α-case. Then η_0 has to be constant and $\mathscr{E}_0 = \overline{0}$.*

Proof. The Killing form is positive definite on the Cartan subalgebra containing η_0. From this it follows that η_0 is constant. From proposition (2.8) one finds that $\mathscr{E}_0$ has to be a subalgebra of this Cartan subalgebra. But there does not exists a decomposition of $\mathscr{L}_0$ into this $\mathscr{E}_0$ and a complementary subalgebra. ∎

2.10. PROPOSITION. *Let V be as in the β-case. Then $\mathscr{E}_0$ has to be a subalgebra of the Borel subalgebra of g constructed upon the negative root spaces and η_0 is constant iff $\mathscr{E}_0$ is in the centraliser of g $_{\beta}$*

The proof follows directly from proposition (2.8).
Similar arguments can be used for further reduction.

(ii) The selection of different decompositions.

Let $\mathscr{L} = n_1 \oplus k_1 = n_2 \oplus k_2$ be two decompositions both satisfying condition (C) and let $n_1 \subsetneq n_2$; one has that $\overline{k}_1^{\perp} \subset \overline{k}_2^{\perp}$. This allows one to consider a submanifold $W \subset \overline{k}_1^{\perp}$ which is coad (n_1) as well as coad (n_2) invariant. The actions can be compared which allows one to define a reduction of a given system.

2.11. PROPOSITION. *Let n_1 and n_2 define two decompositions of $\mathscr{L}$ defined as above and let $n_2 = n_1 + m$, with $n_1 \cap \mathscr{L}_- = n_2 \cap \mathscr{L}_- = \mathscr{L}_-$. The system defined by n_1 and a momentum operator σ is a reduction of the system defined by n_2 if* $p_{k_2} \circ \mathrm{ad}\,(m)\,(k^{\perp 1}) = 0$.

REMARK. If m is an ideal in g, in case g is a more general Lie algebra, one has $p_{k_2} \circ \mathrm{ad}\,(m) = 0$ and the condition is satisfied trivially.

To prove this proposition it suffices to write down the flow equations explicitly.

(iii) Transformations of the momentum operator.

Let $\sigma : J \to \overline{k}^{\perp}$ be a holonomic momentum operator and G a transformation group acting on $\overline{k}^{\perp}$ by conjugation $y \longmapsto g \cdot y \cdot g^{-1}$. If $\nu : J \to G$ is a smooth map and $g \in G$ we denote $\nu^* g$ by $^*g$. Any momentum operator transforms under this action as $\sigma \longmapsto \tilde{\sigma} = {}^*g \cdot \sigma \cdot {}^*g^{-1}$, while an evolution equation for σ becomes

$$D_{t_i}\tilde{\sigma} = [D_{t_i}{}^*g \cdot {}^*g^{-1}, \tilde{\sigma}] + {}^*g \cdot D_{t_i}\sigma \cdot {}^*g^{-1}.$$

2.12. PROPOSITION. *Let $\Psi : J \to k^1$ be a smooth map and let $\sigma : J \to \overline{k}^{\perp}$ be an holonomic momentum operator on $S : \sigma^* p_k \nabla H_0 = \Psi$. Then $\tilde{\sigma} = {}^*g\sigma{}^*g^{-1}$ is an holonomic momentum operator on $\tilde{S} : {}^*g \cdot \sigma^* p_k \nabla H_0 \cdot {}^*g^{-1} + D_n{}^*g \cdot {}^*g^{-1} = {}^*g \cdot \Psi \cdot {}^*g^{-1}$, for any map $\nu : J \longmapsto G$.*

REMARK. (1) Any G leaving $\overline{k}^{\perp}$ invariant has to be a group with Lie algebra $\{\sum_{-\infty}^{0} \xi_i \lambda^i \mid \xi_i \text{ in } g\}$. In general one chooses elements of the group with Lie algebra $\mathscr{L}^0$, and more generally those elements leaving Ψ invariant.

(2) Neither the space k^1 nor the space n has to be invariant under G.

(3) This proposition also allows one to enlarge a given system of evolution

equations by choosing ν on a larger jet bundle.

Proof. It is sufficient to observe that tangency of the flows is conserved under this type of transformation. The truncating equations transform accordingly while the constraint equation transforms as given in the propostion. ■

2.13. PROPOSITION. *Let $\Psi : J \to k^1$ be defined as before and let K be the subgroup of $G(\mathscr{L}_0)$ leaving $\partial_\lambda \Psi$ invariant. Then it follows from the* ad-*invariance of the forms $(\ ,\ )_m$ that the conservation laws* (2.3(c)) *are invariant under K.*

If in addition $\partial_\lambda \Psi$ is constant then the conservation laws are a subset of the Backlund equations. Hence one may look for a reduction to select the conservation laws out of the Backlund equations. We will show that this is not always possible in the examples. In the case β one obtains in this way the systems given by V.G. Drinfeld and V.V. Sokolov [7].

As a final remark one sees that if ν is taken constant one is able to construct parameter families of complete integrable systems. We will show as an example how the 1-parameter family given by *B.* Kuperschmidt relating MKdV and KdV is obtained in this way. This situation is very interesting because the integration of the flows on the Lie algebra is the same for all members of the family.

3. CONSTRUCTION OF MOMENTUM OPERATORS

The construction of a momentum operator is central for the construction of completely integrable systems as well as for the understanding of their structure. This paragraph is based upon two basic constructions for a simple Lie algebra. Both constructions are then easily adapted to other decompositions. This will be shown for some particular cases arising from the decomposition appearing in [2].

Let g be a simple Lie algebra of rank ℓ and Φ a root system for g. $\{\alpha_i\}$ is the set of simple roots, β the highest root and h the Cartan subalgebra. With respect to the corresponding rootspace decomposition of g, b will be the Borel subalgebra constructed on the negative rootsubspace. The natural grading given by the heights of the roots will be indicated by means of the parameter μ. Then for $\xi \subset g$ one has $\xi = \Sigma\, \xi_a \mu^a$. We recall that there exist ℓ traceforms $\{Q_i\}$, which generate the ad-invariant polynomials on g.

The two next lemmas are formulated on the Kac-Moody-Lie algebra constructed from g. The first lemma is due to G. Wilson. It will be presented without proof. Because we are interested in holonomic momentum operators we want to have a very partical solution of the given equations. This is the reason why the lemmas are presented in this specific form.

3.1. LEMMA. (G. Wilson [22]). *Let* $\Psi = \psi_0 + \lambda \psi_1$, *with* $\psi_0, \psi_1 : J \to g$, *smooth maps, J any jet bundle,* ψ_1 *constant and regular in h and* ψ_0 *with values in* $h^\perp$. *Then there exists a* $\sigma : J \to \mathscr{L}^0$ *uniquely defined by the set of constants* $\{E_{ij} \mid i = 1, \ldots, \ell; j = 1, \ldots, \infty\}$ *and the requirements*

(a) $\sigma_0 = \psi_1$

(b) $Q_i(\sigma) = \sum_{j=0}^{\infty} E_{ij} \lambda^{-j}$, *with* $E_{i0} = Q_i(\psi_1)$

(c) $D_x \sigma = [\Psi, \sigma]$.

REMARK. Using the fact that any $\sigma : J \to \mathscr{L}^0$ solving (c) has to have its highest element in h, it is easy to select the solution given in this lemma from the set of solutions given by G. Wilson. As it is indicated by G. Wilson one can generalise this lemma to those Ψ with ψ_1 constant in any Cartan subalgebra of g.

3.2. LEMMA. *Let* $\Psi = \psi_0 + \lambda \psi_1$, *with* $\psi_0, \psi_1 : J \to g$, *smooth maps, J any jet-bundle,* ψ_1 *constant in* $g_{-\beta}$ *and* $\psi_0 = B + \Sigma \exp \phi_i \cdot e_{\alpha_i} \cdot B$ *takes values in b. If rank* $g = 2$ *one requires that* $(\psi_0 \cdot \psi_1) = 1$. *Then there exists a unique solution* $\sigma : J \to \mathscr{L}^0$, *defined by the set of constants* $\{E_{ij} \mid i = 1, \ldots, \ell; j = 1, \ldots, \infty\}$ *and the requirement*

(a) $\sigma_0 = \psi_1$

(b) $Q_i(\sigma) = \sum_{j=0}^{\infty} E_{ij} \lambda^{-j}, E_{i0} = 0, E_{i1} = 0, i \neq \ell$

(c) $D_x \sigma = [\Psi, \sigma]$.

This lemma will be proven by means of the following statements, which constituite the construction of the solution. One observes that the set of solutions is a linear space. Hence it is sufficient to prove the existence for $E_{i1} \neq 0$ and $E_{ij} = 0$, $j \geqslant 2$.

STATEMENT 1. Each equation

$$D_x \xi_a = [\psi_0, \xi]_a,$$

with $\xi \in g$ and a the grading index of g, contains only ξ-terms of height $\geqslant a - 1$. Moreover each element of height $a - 1$ is multiplied by a non negative element (of height 1) in ψ_0.

STATEMENT 2. There exists a partition B, containing ℓ classes, of the set of root subspaces and base spaces of h (with respect to Φ), such that each class of B is an ordered set. Moreover the partition B itself is an ordered set, the ordering being given by the height of the largest element of each class.

Let $\alpha = 1, \ldots, \ell$ be the class index, which is taken in agreement with this ordering; the coordinates of the highest (resp. lowest) elements of the classes are denoted by $({}^{+}\xi^{\alpha})$ (resp. $({}^{-}\xi^{\alpha})$).

STATEMENT 3. The set of equations

(1) $$D_x \xi = [\psi_0, \xi]$$

is equivalent to

(2) $$\xi^i = P^i({}^{+}\xi^{\alpha})$$

and the ℓ remaining equations

(3) $$D_x({}^{-}\xi^{\alpha}) = {}^{-}([\psi_0, \xi])^{\alpha}.$$

The P^i are differential operators in D_x, with coefficients in $\mathcal{F}(J)$. Substitution of (2) in (3) gives the recursion relations on the set $({}^{+}\xi^{\alpha})$.

STATEMENT 4. The equations $D_x Q_i(\xi) = 0$, $i = 1, \ldots, \ell$, are necessary conditions for the equations (1).

STATEMENT 5. Consider the equations

(4) $$D_x \xi = [\psi_0 + \lambda \psi_1, \xi], \quad \xi \in \mathcal{L}^0.$$

(a) The equations (4) reduce to

(5) $$\xi^i = P^i_\lambda({}^{+}\xi^{\alpha})$$

and the ℓ recursion relations

(6) $$R_\alpha({}^{+}\xi^{\beta}) = 0.$$

(b) For all ξ^i of positive and zero height, the corresponding operators P^i are independent of λ.

(c) For all ξ^i of negative height, the corresponding operators P^i_λ are linear in λ.

(d) The image of the linear part in λ of each operator P^i_λ, of height $-a$, is a polynomial over $\mathcal{F}(J)$, which is linear in the variables $D_x^{r}\,{}^{+}\xi^{\alpha}$, $0 \leqslant r < a$, $d - a + r - 1 < m_\alpha \leqslant d - 1$, where d is the Coxeternumber and m_α the exponent correspondig with the ordered class α'

(e) Substitution in $\lambda Q_\ell(\xi) =$ constant and $Q_i(\xi) = 0$, $i \neq \ell$, of (5) together with

$$^{+}\xi^{1} = \sum_{i=0}^{\infty} A_i \lambda^{-i}, \quad A_0 \neq 0$$

$$^{+}\xi^{\alpha} = \sum_{i=1}^{\infty} B^{\alpha}_{-i} \lambda^{-i}, \quad \alpha \neq 1,$$

gives the unique solution $\sigma : J \to \mathscr{L}^0$ of lemma (3.2).

The proofs of the statements 1 to 4 are straightforward while the proof of statement 5 (specially (e)) requires an explicit use of the ad-invariant traceforms for the different simple Lie algebras.

Both lemmas will be used to construct holonomic momentum operators for some specific decompositions.

According to the work of M. Adler and P. Van Moerbeke we distinguish two basic types of decompositions.

(1) The spinning top types

Here $\mathscr{L} = n \oplus k$ satisfies condition (C) and $k \cap \mathscr{L}_+ = \mathscr{L}_+$.
Such decompositions contain the decompositions considered by W. Symes, determined by parabolic subalgebras of g [20].

(2) The Toda types

Consider the algebra $\mathscr{L}'$ which is obtained from $\mathscr{L}$ by the condition $\lambda = \mu^d$, where d is the Coxeter number of g. Let τ be the Cartan conjugation given by

$$\tau : \mathscr{L}' \to \mathscr{L}'$$
$$(\xi_i \mu^i) \to (\xi_i \mu^{-i}).$$

Let $n = \mathscr{L}_- \oplus \mathscr{E}_0$, with $\mathscr{E}_0 = b$, and define

$$k = \{\xi \in \mathscr{L}' \mid \xi = -\tau(\xi)\}.$$

This decomposition satisfies condition (C).

Both types of decomposition are discussed shortly and it will be shown how the two lemmas are applied.

1. The spinning top types (with $n = \mathscr{L}_-$)

(a) η_0 is constant.

α-case: $\eta_0 = e_{-\beta}$.

Because $\dot{\eta}_{-1} = [\eta_0, \varphi_{-1}]$, with φ_{-1} the component in $\mathscr{L}^{-1}/\mathscr{L}^{-2}$ of $p_n \nabla H$, for an $H \in \mathscr{F}(\overline{k}^{\perp})$, one finds that $\eta_{-1} \subset \mathrm{Imad}\,(x_0)$. This implies that W defined by $\eta_0 = e_{-\beta}$, $\eta_{-1} = \gamma + \sum e_{\alpha_i}$, where γ are the coordinates on the Borelsubalgebra b, is coad (n)-invariant.

The commuting flows determined by the quadratic forms in $\mathscr{A}(W)$ are determined by the gradients:

$$p_k \nabla H_r = \eta_{-r} + \lambda\, \eta_{-r+1} + \ldots + \lambda^r \eta_0 .$$

Let Ψ be as in lemma (3.2) and $\sigma : J \to \overline{k}^{\perp}$ a solution of the equation $D_x \sigma = [\Psi, \sigma]$. The constraint equation using H_0 becomes

$$S : \sigma_{-1} = \psi_0 .$$

The following properties are easily verified.

3.3. PROPERTIES. (1) Equation S implies that $\phi_i = 0$, $\forall i$ (the ϕ_i determined as in lemma 3.2).

(2) The components of the equation $\sigma_{-1} = \psi_0$ which lie in the Cartan subalgebra are identities.

(3) If Ψ is a smooth free map and if B is of maximal rank, then any σ as in lemma (3.2) is a momentum operator which is holonomic on S.

One observes that the decomposition in invariant under the adjoint action of $G(\mathscr{L}_0)$. Moreover the subgroup generated by the nilpotent subalgebra in b leaves the manifold Q invariant. The constraint equations become

$$\psi_0 + D_x g \cdot g^{-1} = \sigma_{-1} .$$

Using the reduction procedure as defined above one obtains a representation of $\mathfrak{K}$ equations. Any transformation from one representation to another one is given by a Miura-type transformation (see V. Drinfeld, V. Sokolov). Bearing in mind that the conservation laws are invariant under this type of transformation, this set does contain the conservation laws.

Further reductions of these systems are obtained from a decomposition with $\mathscr{E}_0 \neq 0$, such that $[\mathscr{E}_0, \psi_1] = 0$, or by $\mathscr{E}_{-1} \neq \mathscr{L}^{-1}/\mathscr{L}^{-2}$, or by the use of transformations in $G(\mathscr{L}_-)$.

β-case

$\eta_0 = h_{\alpha_i}$, with h_{α_i} regular in h.

In this case one finds that $\dot{\eta}_0 \subset \mathrm{Imad}\,(\eta_0)$. This allows us to define W by $\eta_0 = h_{\alpha_i}$ and $(X, \eta_{-1}) = 0$, $\forall X \in h$.

The following properties are easily verified.

3.4. PROPERTIES. Let Ψ and σ be as in lemma (3.1) and let

$$S : \sigma_{-1} = \psi_0.$$

Then

(1) S is satisfied identically

(2) if Ψ is a smooth free map and if ψ_0 with values in Im (ad h) is of maximal rank, then any σ as in lemma (3.1) is a holonomic momentum operator.

The subgroup of $G(\mathscr{L}_0)$ leaving h_{α_i} invariant is the ℓ-dimensional abelian subgroup generated by h. But the situation is less rich than in the former case.

As a consequence of proposition (2.9) no reductions by choosing $\mathscr{E}_0 \neq 0$ are possible.

(b) η_0 is non constant.

Following proposition (2.9) one is only left with case β.

Let $\eta_0 = e^f e_{-\beta}$. This implies that $\mathscr{E}_0 \subset b$ with $\mathscr{E}_0 \cap h \neq \phi$.

Lemma (3.2) generalises to this case and with Ψ a free smooth map satisfying the rank conditions one finds σ to be a momentum operator which is holonomic on $S : \sigma_{-1} = \psi_0$.

Reductions of these systems are constructed in a similar fashion to the above.

2. The Toda types

Let $\mathscr{L} = n \oplus k$ be a Toda type decomposition; then $k^\perp$ are the symmetric elements in $\mathscr{L}'$.

The α-case is excluded as follows from proposition (2.9).

Let η_0 be in the space $g_{-\beta}$, then $\dot{\eta}_0 \neq 0$ for some Hamiltonian flow. Lemma (3.2) generalises to the following case:

Let $\Psi = -\lambda^{-1}\psi_1^\tau + \psi_0 - \psi_0^\tau + \lambda\psi_1$ with $\psi_1 = e^f e_{-\beta}$ and $\psi_0 = \Sigma \exp \phi_i \cdot e_{\alpha_i}$. Then, together with similar conditions as in (3.2), there exists a solution σ of the equation $D_x\sigma = [\Psi, \sigma]$, $\sigma : J \to \mathscr{L}^0$, whit $\sigma_0 = \psi_1$. If in addition the set $\{f, \phi_i\}$ is a free set of maximal rank and if rank $g > 2$, then σ is a momentum operator. If the Lie algebra g is an A_1 algebra, then the condition $(\psi_0, \psi_1) =$ constant implies that $\phi = -f$. Any non constant function f defines a momentum operator.

REMARKS AND GENERALISATIONS. (1) The above cases show already how the two lemmas have to be generalised to cover other types of decompositions. More specific choices for Ψ as for example $\Psi = \sum_{i=-p}^{1} \psi_i \lambda^i$ are possible as

long as the highest terms remain as in the lemmas and the ψ_i are determined on a given decomposition $n \oplus k$.
Also more terms in the positive powers are possible. If $n = \mathscr{L}^0$ one sets $\Psi = \lambda \cdot \psi_1$. The Backlund equations themselves now becomes flows of a coadjoint action.

(2) In order to describe systems in more variables, the above constructions may be generalized in the following way.
Let $g = \sum_{i=0}^{p} g_i a^i$, with $a^{i+1} = 0$, 1 or -1, and g any simple (real) Lie algebra. One then constructs the Kac-Moody-Lie algebra over g. The equation defining the momentum operator is now given by

$$L\sigma = [\Psi, \sigma]$$

with

$$L = \sum_{i=0}^{p} a^i D_{x^i},$$

and $\Psi : J \to k^1$. J is now a jetbundle of elements in $C^\infty(\mathbb{R}^p, \mathbb{R}^m)$. The choice of Ψ has to be in agreement with some generalised formulation of the lemas. Examples will be given in the next paragraph.
More exotic operators (including projection operators) can be taken for L. The main point being that the variables (x^i) have to intertwine with the Lie algebra structure.

(3) The reductions obtained from $G(\mathscr{L}'_-)$ are studied by V. Drinfeld - V. Sokolov [7]. They also give the main lemma to deal with these reductions.

(4) G. Wilson [22] showed that there exists a transformation in the Lie algebra $\mathscr{L}$ relating a special reduction of case α with a special reduction of case β. This class contain the systems described by B. Kuperschmidt and G. Wilson [14] [15] using the cyclic elements in $g\ell(n, \mathbb{C})$, the *TD*-systems described by G. Wilson [22] and the systems described by V. Drinfel'd and V. Sokolov [7]. We recall that the class of Toda-type systems we consider do not contain the 2 *TD*-sytems defined by G. Wilson.

4. EXAMPLES

(a) Examples on $sl(2, \mathbb{R})$.

Let (e_1, e_2, e_3) be a base for $sl(2, \mathbb{R})$ such that $[e_1, e_2] = e_1$, $[e_1, e_3] = 2e_2$,

$[e_2, e_3] = e_3$. The coordinates on J are $(x, u, v, u_x, v_x, u_{xx}, v_{xx}, \ldots)$ and the map Ψ is defined by $\Psi = (ue_1 + ve_2 - e_3) + \lambda e_1$.
The solution of $D_x\sigma = [\Psi, \sigma]$, as given by lemma (3.2) is defined as follows. Let $\xi^3 = \sum_{i=1}^{\infty} A_{-i}\lambda^{-i}$; then one has

$$\xi^1_{-k} = \frac{1}{2}(D_x^2 - D_x \cdot v - 2u)A_{-k} - A_{-k-1}$$

$$\xi^2_{-k} = (D_x - v)A_{-k}.$$

The remaining equation is

$$(D_x^3 + vD_x^2 - D_x^2 v - 2D_x u - 2uD_x - vD_x v)\,\xi^3 = 4\lambda D_x \xi^3,$$

which is solved by means of the Killing form:

$$4\lambda\xi^3\xi^3 + (4u + 2v_x + v^2)\xi^3\xi^3 + D\xi^3 D\xi^3 - 2\xi^3 D^2\xi^3 = 4\lambda^{-1} + E_{-2}\lambda^{-2} + \ldots$$

where the E_{-i}'s are constants fixing the orbit. Substitution of ξ^3 with $A_{-1} = 1$ yields

$$A_{-2} = \frac{1}{8}[E_{-2} - 4u - 2v_x - v^2]$$

$$A_{-3} = \frac{1}{8}E_{-3} + \frac{1}{4}[D_x^2 - E_{-2} + 6A_{-2}]A_{-2}$$

This defines σ completely. The C^i are given by $A_{-i} = 0$.

Example (1)

$n = \mathscr{L}_-$, $\quad k = \mathscr{L}_0 \oplus \mathscr{L}_+$, $\quad k^\perp = \mathscr{L}_+$, $\quad W = \{\eta \in \overline{k}^\perp \mid \eta_0 = -e_1, (\eta_0, \eta_{-1}) = 1\}$.

The elements in $\mathscr{A}(W)$ for a given $p \in \mathbb{N}$ and $W \subset k^{\perp p}$ are generated by:

$$H_{-p-1} = (\xi_p, \xi_p)$$

$$H_{-p} = 2(\xi_p, \xi_{p-1})$$

$$\ldots$$

$$H_{-1} = 2(\xi_p, \xi_0) + 2(\xi_{p-1}, \xi_1) + \ldots$$

$$H_0 = 2(\xi_p, \xi_{-1}) + 2(\xi_{p-1}, \xi_0) + \ldots$$

$$\ldots$$

$$H_{p-3} = 2(\xi_p, \xi_{-p+2}) + 2(\xi_{p-1}, \xi_{-p+3}) + \ldots + 2(\xi_2, \xi_0).$$

It is easily found that the flows of $H_{-p-1}, \ldots, H_{-2}$ are trivial on W, while the other flows at each point $p \in W$ span the tangent space of a Lagrangian manifold of the orbit through p of the coadjoint action of $G(n)$. In the inverse limit structure the Hamiltonian vectorfields are defined by the gradients:

$$p_k \nabla H_{i-1} = \eta_{-i} + \lambda \eta_{-i+1} + \ldots + \lambda^i \eta_0 .$$

According to theorem (2.3) any flow

$$D_{t_i} \sigma = [\sigma^* p_k \nabla H_i, \sigma]$$

is determined by $D_{t_i} \psi_0$. The Bäcklund equation gives

$$u_t = (D - v) A_{-i-1}$$

$$v_t = 2A_{-i-1} .$$

The constraint equation becomes

$$S : 0 = v_x + \frac{1}{2} (E_{-2} - 4u - v^2).$$

The first non trivial flow is

$$v_t = -\frac{1}{8} (v_{xxx} - 3 v_x v_x - E_{-2} v_x) + \frac{1}{8} E_{-3} .$$

The systems are completely integrable for each term in the inverse limit. This is checked by defining the dimension of on $\mathcal{O}_x$ in each space $k^{\perp p}$.

Example (2)

Let $\beta \in \mathbb{R}$ and define the 1-parameter family of decompositions by

$$n = \mathcal{L}_- \oplus \mathbb{R} e_1$$

$$k = \mathbb{R}(e_3 - \beta e_2 + \beta^2/_4 e_1) \oplus \mathbb{R}(e_2 - \beta/_2 e_1) \oplus \mathcal{L}_+$$

$$k^\perp = \mathbb{R}(e_3 - \beta e_2 + \beta^2/_4 e_1) \oplus \mathcal{L}_+ .$$

which is defined by a null rotation in $E^{1,2} \cong sl(2, \mathbb{R})$. Projection upon k is defined by

$$P_k(ae_1 + be_2 + ce_3) = (b + c\beta)(e_2 - \beta/_2 e_1) + c(e_3 - \beta e_2 + \beta^2/_4 e_1).$$

The Hamiltonians $H_{-p-1}, \ldots, H_{-2}$ are trivial on $W \subset k^{\perp p}$ and the constraint equation is

$$S : u = -\beta/_2 v + \beta^2/4.$$

The flows are, for $E_i = 0, i \geqslant 2$.

$$v_{t_i} = D(D - v + \beta) A_{-i},$$

which gives for $i = 2$

$$v_t = -\frac{1}{4}\left(v_{xxx} - \frac{3}{2} v^2 v_x + 3\beta v v_x - \frac{3}{2}\beta^2 v_x\right).$$

This is the 1-parameter family due to B. Kupershmidt [12].

Example (3)

Consider the decomposition $n = \mathscr{L}_- \oplus \mathbb{R} e_1 \oplus \mathbb{R} e_2$ and $k = \mathscr{L}_+ \oplus \mathbb{R} e_3$. It is sufficient to use

$$\Psi = e^{-f} e_3 + \lambda e^f e_1$$

as is easily seen from the form of k.

This gives from $D_x \sigma = [\Psi, \sigma]$

$$\zeta^1 = \frac{1}{2}(e^{2f} \cdot f_x D_x + e^{2f} D_x^2 + 2\lambda e^{2f})\, \zeta^3$$

$$\zeta^2 = -e^f D_x \zeta^3.$$

Constancy of the Killing form gives, with $E_i = 0, i \geqslant 2$.

$$A_{-1} = e^{-f}$$

$$A_{-2} = \frac{1}{2} e^{-f}(2 f_{xx} + f_x \cdot f_x)$$

etc.

and $\zeta^3 = \sum_{i=1}^{\infty} A_{-i} \lambda^{-i}$.

The flows are given by $D_{t_i} e^{-f} = D_x A_{-i}$. The first nontrivial equation is

$$\dot{f} = -\frac{1}{8}(2 f_{xxx} - f_x f_x f_x).$$

One remarks that the constraint equation S is trivially satisfied.

Example (4)

Consider $n = \mathscr{L}^{-2} \oplus (\mathbb{R} e_2 \oplus \mathbb{R} e_1)\lambda^{-1} \oplus \mathbb{R} \cdot e_1$.

$$k = \mathbb{R} e_3 \lambda^{-1} \oplus (\mathbb{R} e_3 \oplus \mathbb{R} e_2) \oplus \mathscr{L}_+ .$$

The submanifold W is defined by $\eta_0 = e_1$, $\eta^3_{-1} = 1$. It is sufficient to write

$$\Psi = \omega e_3 \cdot \lambda^{-1} + (e_3 + v e_2) + e_1 \cdot \lambda.$$

The equation $D_x \zeta = [\Psi, \zeta]$ has a solution with

$$\zeta^1 = \frac{1}{2\alpha}\left[-\frac{\alpha_x}{\alpha}(D_x - v) + \frac{1}{\alpha} D_x(D_x - v) + 2\lambda\right]\zeta^3$$

$$\zeta^2 = -\frac{1}{\alpha}(Dx - v)\,\zeta^3, \quad \alpha = 1 + \lambda^{-1}\omega,$$

and

$$\zeta^3 = \sum_{i=1}^{\infty} A_{-i}\lambda^{-i},\ A_{-1} = 1,\ A_{-2} = \frac{1}{2}\left(\omega + \frac{1}{2}v_x + \frac{1}{4}v^2\right), \ldots,$$

which defines the equation

$$S : \omega = \frac{1}{2}v_x + \frac{1}{4}v^2.$$

The first nontrivial evolution equation is

$$\dot{v} = -\frac{1}{4}(v_{xxx} + v_x v_x - v^2 v_x).$$

Example (5)

Using lemma (3.1) and

$$\Psi = u e_1 + \omega e_3 + \lambda \cdot a e_2 \quad (a = 1 \text{ or } i)$$

Then with $u = \mathscr{L}_-$ and $k = \mathscr{L}_0 \oplus \mathscr{L}_+$ one finds if

(i) $a = 1$ and using the reduction $u = \omega$: the MKdV-equation, or if one sets $\omega = 1$: the KdV-equation

(ii) $a = i$ and using $u = \bar{\omega}$: the nonlinear Schrödinger equation.

In all these cases there is no constraint equation. These examples are now well known.

Example (6)

Using the Toda type decomposition one needs to consider

$$\Psi = e^f e_1 \cdot \lambda^{-1} + e^{-f}(e_1 + e_3) + e^f e_3 \cdot \lambda.$$

Remark that in this basis $\tau(e_1) = -e_3$. The equation $D_x\zeta = [\Psi, \zeta]$ defines

$$\zeta^1 = \left[\frac{1}{2\alpha} D_x \frac{1}{\alpha} D_x + \frac{1}{\alpha}(e^{-f} + \lambda e^f)\right]\zeta^3$$

$$\zeta^2 = -\frac{1}{\alpha} D_x \zeta^3, \quad \text{with} \quad \alpha = e^{-f} + \lambda^{-1} \cdot e^f.$$

Substitution in $Q(\zeta) = -4$ yields with $\zeta^3 = \sum_{i=1}^{\infty} A_{-i}\lambda^{-i}$

$$A_{-1} = e^{-f}, \quad A_{-2} = \frac{1}{2}\left[e^{-f}e^{2f} - e^{-2f} + \frac{1}{4}(2f_{xx} + f_x f_x)\right]$$

$$\cdots \; .$$

Then obviously $\Psi = P_k(\sigma_{-1} + \lambda\sigma_0)$ and hence there is no constraint equation. We set

$$\phi = P_k(\sigma_{-2} + \lambda\sigma_{-1} + \lambda^2\sigma_0),$$

then the Bäcklund equation $D_t\psi - D_x\phi + [\Psi, \phi] = 0$ yields

$$\dot{f} = -\frac{1}{2}\left[3(e^{2f} + e^{-2f}) f_x + \frac{1}{4}(2f_{xxx} + f_x f_x f_x)\right].$$

Example (7)

Let $n = \mathscr{L}^0$, $k = \mathscr{L}_+$ and set

$$\Psi = \lambda(ue_1 + ve_2 + \omega e_3),$$

with $-4u\omega + v^2 = -1$.

The equation $D_x\zeta = [\Psi, \zeta]$ has a solution given by

$$\zeta^1 = \frac{1}{2\lambda\omega}\left[D_x \frac{1}{\lambda\omega}(D_x - \lambda v) + 2\lambda u\right]\zeta^3$$

$$\zeta^2 = -\frac{1}{\lambda\omega}(D_x - \lambda v)\zeta^3, \quad \omega \neq 0,$$

$$\zeta^3 = \sum_{i=0}^{\infty} A_{-i}\lambda^{-i}, \; A_0 = \omega, \; A_{-1} = -\omega v_x + \omega_x v, \ldots .$$

The equations are

$$D_{t_i}\sigma_0 = D_x \sigma_{-i}$$

and the equation S is trivial. The first non-trivial equation is

$$\dot{f} = -v_{xx} + v(f_{xx} + f_x f_x)$$

$$\dot{v} = (-v \cdot v_x + f_x(v^2 - 1))_x$$

with $\omega = e^f$.

One finds that the flows do not span a Lagrangian submanifold of an orbit $\mathcal{C}_x$ in $k^{\perp p}$. Completely integrability follows from a more detailed analysis of the inverse limit structure.

(b) Examples on graded Lie algebras over $s1(2, \mathbb{R})$.

Example (1)

Consider the Lie algebra $g = g_1 + a \cdot g_2$, with $a^2 = 0$ and $g_1 \simeq g_2 \simeq s1(2, \mathbb{R})$.

Let $\Psi = (ue_1 + ve_2 - e_3) + a(fe_1 + ge_2) + \lambda e_1$, then one verifies that the equation $(D_x + aD_y)\zeta = [\psi, \zeta]$ has a solution $\sigma : J \to \mathcal{L}^0$ such that $\sigma_0 = e_1$. Let K be the Killing form on $s1(2, \mathbb{R})$; then

$$K_1 = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

are ad-invariant forms on g. These forms allows one to solve the recurrence equations. If one chooses the decomposition $n = \mathcal{L}_-$, $k = \mathcal{L}_0 + \mathcal{L}_+$, the equation S is

$$u = \frac{1}{2} v_x - \frac{1}{4} v^2$$

$$f = -\frac{1}{2} vg + \frac{1}{2} g_x + \frac{1}{2} v_y$$

and the first non trivial equation is

$$\dot{v} = -\frac{1}{4} v_{xxx} + \frac{1}{2} v_x v_x$$

$$\dot{g} = -\frac{1}{4} g_{xxx} - \frac{3}{4} v_{xxy} - v_x(g_x + v_y).$$

Remark that $a^2 = 1$ yields nothing new because then the Lie algebra decomposes into $g_1 + g_2$ and $g_1 - g_2$.

Example (2)

Consider $g = g_1 + ag_2 + a^2 g_3$ with $a^3 = 1$ and $g_1 \simeq g_2 \simeq g_3 \simeq s1(2, \mathbb{R})$.

Let $\psi = (u, v, -1)^\tau + a(f, g, 0)^\tau + a^2(p, q, 0)^\tau + \lambda(1, 0, 0)^\tau$ and let $\overline{\zeta} = \zeta + a\eta + a^2\rho$ be coordinates on $\mathscr{L}$. The equation $(D_x + aD_y + a^2 D_z)\overline{\zeta} = [\psi, \overline{\zeta}]$ has a solution $\sigma : J \to \mathscr{L}^0$, which is found using the ad-invariant forms

$$K_1 = \begin{bmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & K \end{bmatrix}, \; K_2 = \begin{bmatrix} 0 & 0 & K \\ 0 & K & 0 \\ K & 0 & 0 \end{bmatrix}, \; K_3 = \begin{bmatrix} K & 0 & 0 \\ 0 & 0 & K \\ 0 & K & 0 \end{bmatrix}.$$

Let $\overline{e_i} = e_i + ae_i + a^2 e_i$ and the 1-parameter family

$$\tilde{e}_1 = \overline{e}_1$$

$$\tilde{e}_2 = \overline{e}_2 - \frac{1}{2}\beta\overline{e}_1$$

$$\tilde{e}_3 = \overline{e}_3 - \beta\overline{e}_2 + \beta^2/_4\overline{e}_1, \quad \beta \in \mathbb{R}.$$

We define the splitting

$$n = \mathscr{L}_- \oplus \mathbb{R} \cdot \tilde{e}_1, \quad k = \mathscr{L}_+ \oplus \mathbb{R}\,\tilde{e}_2 \oplus \mathbb{R}\,\tilde{e}_3,$$

which yields the constraint equations

$$u = \frac{1}{4}\beta^2 - \frac{1}{2}\beta \cdot v$$

$$S : f = -\frac{1}{2}\beta \cdot g$$

$$p = -\frac{1}{2}\beta \cdot q.$$

The flows are given by

$$\dot{v} = P_1(A_{-i}) + P_2(B_{-i}) + P_3(E_{-i})$$

$$\dot{g} = P_1(B_{-i}) + P_2(E_{-i}) + P_3(A_{-i})$$

$$\dot{q} = P_1(E_{-i}) + P_2(A_{-i}) + P_3(B_{-i})$$

with

$$\zeta^3=\sum_{i=1}^{\infty} A_{-i}\lambda^{-i},\quad \eta^3=\sum_{i=2}^{\infty} B_{-i}\lambda^{-i},\quad \rho^3=\sum_{i=2}^{\infty} E_{-i}\lambda^{-i},\ \text{and}\ A_{-1}=1$$

$$A_{-2}=-\frac{1}{8}\left[2\left(D_x+\frac{1}{2}v-\beta\right)v+2\left(D_y+\frac{1}{2}g\right)q+2\left(D_z+\frac{1}{2}q\right)g\right]-\frac{1}{8}\beta^2$$

$$B_{-2}=-\frac{1}{8}\left[2\left(D_x+\frac{1}{2}v-\beta\right)g+2\left(D_y+\frac{1}{2}g\right)v+2\left(D_z+\frac{1}{2}q\right)q\right]$$

$$E_{-2}=-\frac{1}{8}\left[2\left(D_x+\frac{1}{2}v-\beta\right)q+2\left(D_y+\frac{1}{2}g\right)g+2\left(D_z+\frac{1}{2}q\right)v\right].$$

The operators P_i are given by

$$P_1=D_x(D_x-v-\beta)+D_y(D_z-q)+D_z(D_y-g)$$

$$P_2=D_x(D_z-q)+D_y(D_y-g)+D_z(D_x-v-\beta)$$

$$P_3=D_x(D_y-g)+D_y(D_x-v-\beta)+D_z(D_z-q).$$

One observes that on the line $x=y=z$ this equation reduces to the 1-parameter family of example $(a-2)$.

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Addendum. In [24] Flaschka, Newell and Ratiu derive the A.K.N.S. equations from the Kostant-Adler-Symes theorem on the Kac-Moody Lie algebra $\mathscr{L}$ constructed from $sl(2,\mathbb{C})$ and using the specific decomposition of $\mathscr{L}$ into $\mathscr{L}_-$ and $\mathscr{L}_0\oplus\mathscr{L}_+$. These are the equations mentioned in example (5), which have been worked out for more general Lie algebras in [22] and [14]. If one considers the x-parameter along the integral curve of the Hamiltonian vector field determined by $H_0=(\zeta,\zeta)_{p-1}$ on $\mathscr{L}^p$, for a given $p\in\mathbb{N}$, as is done is these examples, the constraint equations are trivially satisfied and hence the momentum operator is holonomic for each orbit. This is no longer true for the other Hamiltonians, which imposes constraint equations on the function space if one integrates the finite zone solutions on the Jacobian associated with the orbit [25].

In [24] the aim of the authors is to determine the link between the Kac-Moody structure with the τ-functions (references are given in [24]). It is however not clear how the results of [24] generalise the other Lie algebras and other systems of P.D.E.'s. The Hamiltonian form, which is worked out in detail in these papers, depends upon the type of decomposition of $\mathscr{L}$ and the type of invariant submanifold $W\subset\bar{k}^\perp$ one chooses in the inverse limit space. It seems that the introduction of the momentum operator clarifies a lot in the relation between the flows on the Kac-Moody Lie algebra and the commuting P.D.E.'s. In particular it determines the connection with the work of Bubrovin [8] a.o. and with the finite zone solutions of some field equations, as is shown in [25].

In [24] a second Lie algebraic interpretation is given of the same equations in terms of the translated-invariants theorem of Kostant. The conservation laws are derived in a less general form as we do here and in [25].

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